Journal of Thermal Analysis and Calorimetry, Vol. 55 (1999) 449-454

OPTIMIZING OF BALANCES OF THE SECOND GENERATION

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Abstract

At the 3rd Conference on Vacuum Microbalance Techniques in Los Angeles in 1962, we suggested the use of the equation of motion of a balance $T=J+\ddot{\alpha}+k\dot{\alpha}+C\alpha$ for the calculation of the unknown torque T, and the measurement for that purpose of the values of all the other quantities in this equation. The present paper discusses the consequences of two sources of error relevant for this method. First, the errors caused in the first and second derivatives of the deflection are considered, deduced from two or three deflection measurements separated by small time intervals. Secondly, the consequences of the errors caused by the uncertainties in the deflection measurements are discussed. Consideration of the two errors together leads to an optimal set of values of parameters for the balance handling.

Keywords: balance, mass determination, weighing errors

Introduction

When a momentum $T(t)$ is measured with a rotating balance, we have to consider the differential equation

$$T(t) = J\ddot{\alpha}(t) + k\dot{\alpha}(t) + C\alpha(t) \tag{1}$$

in which J is the rotational inertia, k is the damping constant and C is the torsion constant.

When the torque $T(t)$ to be measured is independent of time, the solution of this differential equation becomes

$$\alpha(t) = \frac{T}{C} + \hat{\alpha}\cos(\omega t + \varphi)e^{-t/\tau} \tag{2}$$

In Eq. (2), we use

$$\tau = \frac{2J}{k} \tag{3}$$

1418–2874/99/ $ 5.00

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

where τ is the relaxation time and

$$\omega=\sqrt{\frac{C}{J}-\frac{k^2}{4J^2}} \tag{4}$$

where ω is the angular oscillation frequency. Additionally, in Eq. (2) we use two constants which can be calculated from the initial conditions, i.e. $\hat{\alpha}$, the amplitude, and φ, the phase of the oscillation.

Our experience with such a balance includes the measurement of very small magnetic forces where the sensitivity of the balance was limited by its Brownian motion. In those measurements, the values of k and C were small, as were those of $1/\tau$ and ω. The angle α was measured as a function of time. From these data, we calculated $\dot{\alpha}t$ and $\ddot{\alpha}(t)$; as values of J, k and C were known, we could calculate $T(t)$ directly from Eq. (1). We reported on this procedure at the 3rd Conference on Vacuum Microbalance Techniques in Los Angeles [1] and at the 21st Conference in Lyon [2]. We suggested application of the procedure to conventional balances and introduced the term 'balance of the second generation'.

As application to standard balances demands a very fast procedure, we suggested the use of an on-line computer for the calculations. We suggested that the angle α of rotation should be measured at three times:

$$t_2,\ t_1=t_2-\Delta t \quad \text{and} \quad t_3=t_2+\Delta t \tag{5}$$

The velocity $\dot{\alpha}(t)$ and acceleration $\ddot{\alpha}(t)$ are calculated via Eqs (6) and (7).

$$\dot{\alpha}(t_2)\cong\frac{\alpha(t_3)-\alpha(t_1)}{2\Delta t} \tag{6}$$

$$\ddot{\alpha}(t_2)\cong\frac{\alpha(t_3)+\alpha(t_1)-2\alpha(t_2)}{(\Delta t)^2} \tag{7}$$

Formula error

The arithmetic error $\Delta_{ar}T$ is the uncertainty due to the fact that Δt, though small, is not near enough to zero, so

$$\Delta_{ar}T=\left\{J\frac{\alpha(t_2+\Delta t)+\alpha(t_2-\Delta t)-2\alpha(t_2)}{\Delta t^2}+k\frac{\alpha(t_2+\Delta t)-\alpha(t_2-\Delta t)}{2\Delta t}+C\alpha(t_2)\right\}-$$
$$-\{J\ddot{\alpha}(t_2)+k\dot{\alpha}(t_2)+C\alpha(t_2)\} \tag{8}$$

With a Taylor approximation for $\alpha(t+\Delta t)$ and $\alpha(t-\Delta t)$ around $t=t_2$, Eq. (8) leads to

$$\Delta_{ar}T=\frac{J}{12}\Delta t^2\frac{d^4\alpha}{dt^4}+\frac{k}{6}(\Delta t)^2\frac{d^3\alpha}{dt^3} \tag{9}$$

From Eq. (2), we get

$$\frac{d^4\alpha(t)}{dt^4}=\omega^4\hat{\alpha}\cos(\omega t+\varphi)e^{-t/\tau}-4\frac{\omega^3\hat{\alpha}}{\tau}\sin(\omega t+\varphi)e^{-t/\tau}-$$
$$-6\frac{\omega^2\hat{\alpha}}{\tau^2}\cos(\omega t+\varphi)e^{-t/\tau}+4\frac{\omega\hat{\alpha}}{\tau^3}\sin(\omega t+\varphi)e^{-t/\tau}+4\frac{\hat{\alpha}}{\tau^4}\cos(\omega t+\varphi)e^{-t/\tau} \tag{10}$$

For $\omega\tau>>1$, Eq. (10) reduces to

$$\frac{d^4\alpha(t)}{dt^4}=\omega^4\hat{\alpha}\cos(\omega t+\varphi)e^{-t/\tau} \tag{11}$$

Under the same condition, we find

$$\frac{d^3\alpha(t)}{dt^3}=\omega^3\hat{\alpha}\sin(\omega t+\varphi)e^{-t/\tau} \tag{12}$$

Insertion of Eqs (11) and (12) into Eq. (9) leads to

$$\Delta_{ar}T=\frac{J}{12}(\Delta t)^2\omega^4\hat{\alpha}\cos(\omega t+\varphi)e^{-t/\tau}+\frac{k}{6}(\Delta t)^2\omega^3\hat{\alpha}\sin(\omega t+\varphi)e^{-t/\tau} \tag{13}$$

with $\omega\frac{2J}{k}=\omega\tau>>1$, Eq. (13) reduces to

$$\Delta_{ar}T=\frac{J}{12}\omega^4\hat{\alpha}(\Delta t)^2\cos(\omega t+\varphi)e^{-t/\tau} \tag{14}$$

By averaging over all possible values of the cos function for small values of t/τ we get

$$\Delta_{ar}T=\frac{J}{12}\omega^4\hat{\alpha}(\Delta t)^2\frac{\sqrt{2}}{2} \tag{15}$$

The error due to the error in α

The measurement procedure we described [3] involves the measurement of α at three different time: t_1, t_2 and t_3. We presume that the errors $\Delta\alpha_i$ in the measured values α_i at these times are independent of each other. It must be mentioned, however, that there are many conditions where this independence does not hold. For instance, if disturbances are caused by the Brownian motion for values of

Δt much smaller than τ, a relation between the errors in α_1, α_2 and α_3 must be taken into account.

Returning to the independent errors in α_1, α_2 and α_3, from Eqs (6) and (7) for the errors in the derivatives of α, we deduce

$$\Delta(\dot{\alpha}) = \frac{\sqrt{2}}{2\Delta t}\Delta\alpha \quad \text{and} \quad \Delta(\ddot{\alpha}) = \frac{\sqrt{6}}{(\Delta t)^2}\Delta\alpha \tag{16}$$

It follows for the error $\Delta_\alpha T$ caused in the measured moment of force by the error in α that

$$\Delta_\alpha T = \sqrt{\frac{6}{(\Delta t)^4}J^2 + \frac{k^2}{2(\Delta t)^2} + C^2}\ \Delta\alpha \tag{17}$$

For small value of Δt, Eq. (17) reduces to

$$\Delta_\alpha T = \sqrt{6}\frac{J}{(\Delta t)^2}\Delta\alpha \tag{18}$$

The combination of the two errors

A striking difference between the two errors expressed by Eqs (15) and (18) is that the first increases, while the other one decreases with increasing Δt. They equal one another when

$$\Delta t = \Delta_{eq} t = \sqrt[4]{\frac{24\sqrt{3}\,\Delta\alpha}{\hat{\alpha}}}\,\frac{1}{\omega} = 2.54\frac{1}{\omega}\sqrt[4]{\frac{\Delta\alpha}{\hat{\alpha}}} \tag{19}$$

Thus, for values of $\Delta t \gg \Delta_{eq} t$, the error $\Delta_{ar} T$ (Eq. (15)) will predominate, whereas for $\Delta t \ll \Delta_{eq} t$, according to Eq. (19), $\Delta_\alpha T$ will predominate. If $\Delta_{comb} T$ is introduced to express the total uncertainty in T, we may use

$$\Delta_{comb} T = \sqrt{(\Delta_{ar} T)^2 + (\Delta_\alpha T)^2} \tag{20}$$

By simple arithmetic, it can be shown that $\Delta_{comb} T$ has a minimum value when

$$\Delta t = \Delta_{eq} t \tag{21}$$

Comparison with a conventional balance

We shall indicate the characteristic data on a conventional balance used for comparison by means of dashes. This will relate to a slightly damped balance with observations by the reversal point measuring method. This means that the time interval τ_{meas} necessary for a measurement will be somewhat larger than half the oscillation time to ensure that two subsequent reversal points can be observed. We shall take

$$\tau'_{meas} = \tau'_{osc} \tag{22}$$

For this comparison, we shall suppose that this 'reversal point balance' has as mechanical characteristics

$$J' = J \text{ and } k' = k \tag{23}$$

In order to ensure the equality of the time intervals necessary for the measurement, we wish to ensure that

$$\frac{2\pi}{\omega'} = \tau'_{meas} = 2\Delta t \tag{24}$$

This can be achieved by adjusting the value of C', where Eq. (22) is made use of either by adjusting the centre of gravity or by using an extra electronic feedback device.

When k is small (Eq. (4)), for estimation purposes we may use

$$\tau'_{osc} = 2\pi\sqrt{\frac{J'}{C'}} = 2\Delta t \tag{25}$$

or

$$C' = \frac{\pi^2 J}{(\Delta t)^2} \tag{26}$$

To calculate $\Delta'_\alpha T'$, it must be borne in mind that, with the reversal point method, we use

$$\frac{T'}{C'} = \alpha'_0 = \frac{\alpha(t)_{max} + \alpha(t)_{min}}{2} \tag{27}$$

where $\alpha(t)_{max}$ and $\alpha(t)_{min}$ are two subsequent reversal points. Thus, we get approximately

$$\Delta'_\alpha T' = \frac{\sqrt{2}\Delta\alpha C'}{2} \tag{28}$$

or

$$\Delta'_{\alpha}\,T' = \frac{J\pi^2\sqrt{2}\,\Delta\alpha}{2(\Delta t)^2} \tag{29}$$

From a comparison of Eqs (18) and (29) and consideration of the fact that for both equations we know $\tau_{meas}=2\Delta t$ (Eqs (5) and (24)), we find an advantage by a factor of approximately 3 for the balance of the second generation. It might be expected, therefore, that, when a balance is to be constructed, the preference for one of the two methods might be governed by the nature of the intended application.

References

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